On Adjacency and e-Adjacency in General Hypergraphs: Towards a New e-Adjacency Tensor

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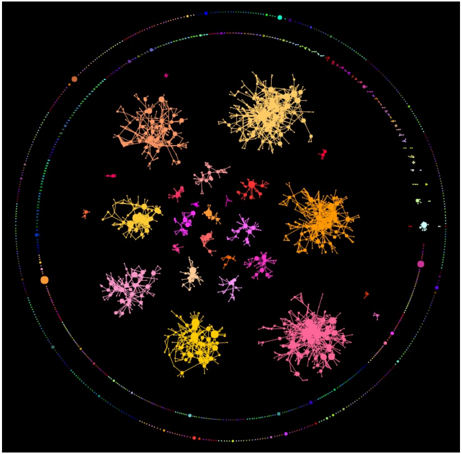
Background

Ideas behind

- Ranking of vertices in graphs
  => **random walks**
- RW for hypergraphs exist
- Diffusion = local process
  => knowledge of the neighbourhood.
- Study of diffusion process => **Laplacian**
- Incidence and adjacency matrices keep only pairwise information
- Pairwise adjacency is too restrictive for hypergraphs
- **Higher order adjacency requires tensor**
- Laplacian tensor is linked to the adjacency tensor
- **Adjacency tensor for uniform hypergraph is known Cooper and Dutle [2012]**



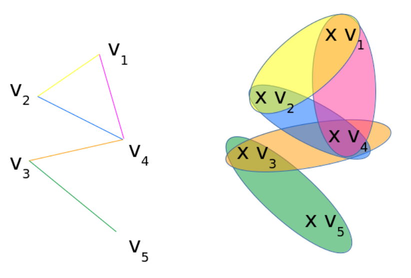
Our contribution

Key points

- Rigourous **definition of adjacency in hypergraphs**
- A proposal for an e-adjacency tensor **interpretable in term of hypergraph uniformisation**
- Two processes are designed:
  - **a hypergraph uniformisation process** (HUP)
  - **a polynomial homogeneisation process** (PUP)

Hypergraphs

From graphs to hypergraphs



- **Hypergraphs** $\equiv$ generalisation of graphs to multiple vertex links
- Hypergraphs introduced by Berge and Minieka [1973].

Definition

Bretto [2013]:
A **hypergraph** $\mathcal{H}$ on a finite set $V = \{v_1 ; v_2; ... ; v_n\}$ is a family of hyperedges $E = \{e_1, e_2, ..., e_p\}$ where each **hyperedge** is a non-empty subset of V.

Two visions

- **set of elements** of power set of vertices $\rightsquigarrow$ set view
- **extension of graphs** $\rightsquigarrow$ n-adic relationship view

k-uniform hypergraph

All its hyperedges have same cardinality k.

On adjacency I

In graphs

- Two vertices are said **adjacent** if it exists an edge linking them
  => **pairwise relationship**
- Vertices incident to one given edge are said **e-adjacent**.
  => also **pairwise relationship**
- **e-adjacency and adjacency are equivalent in graphs**

On adjacency II

Extending to hypergraphs

- k vertices are said **k-adjacent** if it exists a hyperedge that hold them
  => multi-adic relationship
- Vertices of a given hyperedge are said to be **e-adjacent**.
  => multi-adic relationship
- **$\overline{k}$-adjacency**: maximal k-adjacency that can be found in a given hypergraph
- In **k-uniform** hypergraph: **equivalence** $\overline{k}$-adjacency and e-adjacency.
- In **general** hypergraphs: the equivalence **doesn't hold** anymore!

Existing k-adjacency tensor for k-uniform hypergraph

Cooper and Dutle (k-)adjacency tensor

- ([Author's note]: **degree normalized k-)adjacency tensor**: Cooper and Dutle [2012]
  $\mathcal{A} = \left(a_{i_1 \dots i_k}\right)_{1 \leqslant i_1, \dots, i_k \leqslant n}$ such that:

$$a_{i_1 \dots i_k} = \frac{1}{(k-1)!} \begin{cases} 1 & \text{if } \left\{v_{i_1}, \dots, v_{i_k}\right\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

- Allows to retrieve degree of vertices:

$$\deg\left(v_i\right) = \sum_{i_2, \dots, i_k = 1}^{n} a_{i i_2 \dots i_k}.$$

- Allows to have hypergraph spectral theory Qi and Luo [2017]

Tensor for general hypergraphs: the art of filling

What about this?

=> **We need to store additional information**

Existing e-adjacency tensor for general hypergraph

Banerjee e-adjacency tensor

Let $\mathcal{H} = (V, E)$ with $V = \{v_1, v_2, ..., v_n\}$ and family $E = \{e_1, e_2, ..., e_p\}$.
Let $k_{\max} = \max\{|e_i| : e_i \in E\}$ be the maximum cardinality of the family of hyperedges.
The ([Author's note]: **e-) adjacency hypermatrix** of $\mathcal{H}$ written $\mathcal{A}_{\mathcal{H}} = \left(a_{i_1 \ldots i_{k_{\max}}}\right)_{1 \leqslant i_1, \ldots, i_{k_{\max}} \leqslant n}$ is such that for a hyperedge: $e = \left\{v_{l_1}, ..., v_{l_s}\right\}$ of cardinality $s \leqslant k_{\max}$.

$$a_{p_1 \ldots p_{k_{\max}}} = \frac{s}{\alpha}, \text{ where } \alpha = \sum_{\substack{k_1, \ldots, k_s \geqslant 1 \\ \sum k_i = k_{max}}} \frac{k_{\max}!}{k_1! \ldots k_s!}$$

with $p_1, ..., p_{k_{\max}}$ chosen in all possible way from $\{l_1, ..., l_s\}$ with at least once from each element of $\{l_1, ..., l_s\}$.

Why an other proposal?

Motivations

- The e-adjacency tensor should be easily interpretable:
  - in term of e- and k-adjacency
  - in term of the way it is built
- Information on k-adjacency should be easy to gather
- Can we really fill a tensor without transforming its spectra?

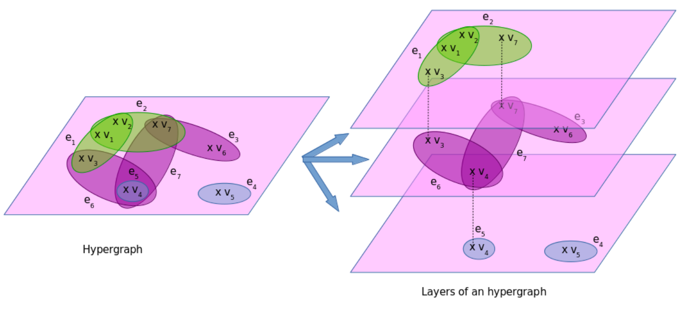
Requirements

The tensor should be:

- invariant to vertex permutations either globally or at least locally.
- allow the retrieval of the hypergraph in its original form.
- the sparsest possible in between two possible choices.
- allow the retrieval of the degrees of the nodes
- store the information of e-adjacency and k-adjacency

Layers I

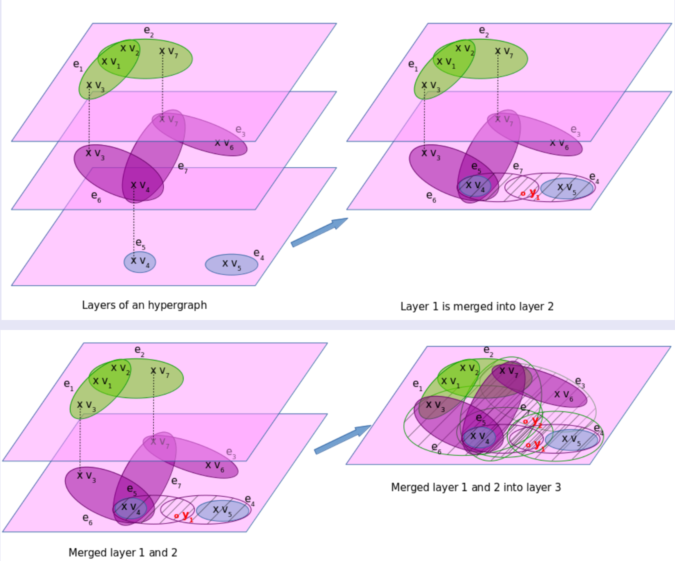
Decomposition of the hypergraph in layers



- $\mathcal{H} = \bigoplus_{k=1}^{k_{\max}} \mathcal{H}_k$, $\mathcal{H}$ is with no repeated hyperedge
- Family of Cooper and Duttle k-adjacency tensors $(\mathcal{A}_k)$: $\mathcal{H}_k \leftrightsquigarrow \mathcal{A}_k$

Filling and merging

Iterative process on layers



Layers of an hypergraph

Layer 1 is merged into layer 2

Merged layer 1 and 2

Merged layer 1 and 2 into layer 3

Hypergraph uniformisation process

Two elementary operations

- y**-vertex-augmentation operation**:
  - add a vertex to each hyperedge of a given hypergraph
  - y**-vertex-augmented hypergraph** $\overline{\mathcal{H}_{\overline{w}}} = \left(\overline{V}, \overline{E}, \overline{w}\right)$ of $\mathcal{H}_w$
- **merging operation**:
  - merges two weighted hypergraphs $\mathcal{H}_a = (V_a, E_a, w_a)$ and $\mathcal{H}_b = (V_b, E_b, w_b)$
  - obtained: **merged hypergraph** $\widehat{\mathcal{H}_{\widehat{w}}} = \left(\widehat{V}, \widehat{E}, \widehat{w}\right)$

The hypergraph uniformisation process

- Transform each $\mathcal{H}_k$ into a weighted hypergraph $\mathcal{H}_{w_k,k} = (V, E_k, w_k)$ $w_k(e) = c_k$
  => **dilatation coefficient**: keep the generalized hand-shake lemma
- Iterates over a two-phase step:
  - **the inflation phase**
  - **the merging phase**

Moving to homogeneous polynomial

Symmetric hypermatrices and homogeneous polynomials

- Symmetric cubical hypermatrices are bijectively mapped to homogeneous polynomials Comon et al. [2015]
- Use the hypermatrix multilinear matrix multiplication Lim [2013].
- $\mathcal{H}_k$ =>$\boldsymbol{A}_k = \left(a_{(k)\, i_1 \ldots i_k}\right)$
- $(\boldsymbol{z})_{[k]} = (\boldsymbol{z}, \ldots, \boldsymbol{z}) \in (\mathbb{R}^n)^k$, $(\boldsymbol{z})_{[k]} . \boldsymbol{A_k}$ contains only one element:
  $P_k(\boldsymbol{z}_0) = \sum\limits_{1 \leqslant i_1, \ldots, i_k \leqslant n} a_{(k)\, i_1 \ldots i_k} z^{i_1} \ldots z^{i_k}.$
- As $\boldsymbol{A_k}$ is symmetric: $P_k(\boldsymbol{z}_0) = \sum\limits_{1 \leqslant i_1 \leqslant \ldots \leqslant i_k \leqslant n} \alpha_{(k)\, i_1 \ldots i_k} z^{i_1} \ldots z^{i_k}$ with
  $\alpha_{(k)\, i_1 \ldots i_k} = k! a_{(k)\, i_1 \ldots i_k}.$

Polynomial uniformisation process I

Principle

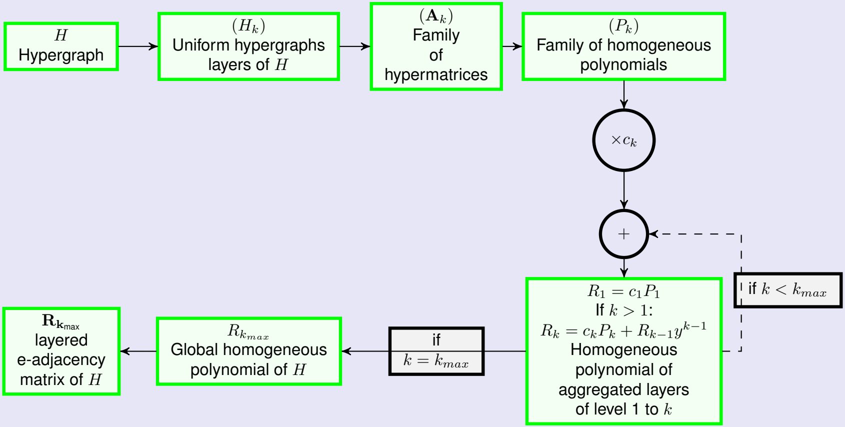


Figure 1: Different phases of the construction of the e-adjacency tensor

Polynomial uniformisation process II

Details

$$R_{k+1}(\boldsymbol{z}_k) = y^{k\,(k+1)}\left(R_k\left(\frac{\boldsymbol{z}_{k-1}}{y^{k\,(k)}}\right) + c_{k+1}P_{k+1}\left(\frac{\boldsymbol{z}_o}{y^{k\,(k+1)}}\right)\right)$$

$$= R_k(\boldsymbol{z}_{k-1})\,y^k + c_{k+1}\sum_{i_1,\ldots,i_{k+1}=1}^{n} a_{(k+1)\,i_1\,\ldots\,i_{k+1}}z^{i_1}\ldots z^{i_{k+1}}$$

From homogeneous polynomial to hypermatrix

Definition of the hypermatrix of layer of level k

$R_k\left(\boldsymbol{w}_{(k)}\right) = \sum_{i_1,\ldots,i_k=1}^{n+k-1} r_{(k)\, i_1\, \ldots\, i_k} w_{(k)}^{i_1} \ldots w_{(k)}^{i_k}$ where:

- for $i \in [\![n]\!]$: $w_{(k)}^{i} = z^i$ and for $i \in [\![n+1; n+k-1]\!]$: $w_{(k)}^{i} = y^{i-n}$
- for all $\forall j \in [\![k]\!]$, for $1 \leqslant i_1 < \ldots < i_j \leqslant n$, for all $l \in [\![j+1; k]\!]$[1]: $i_l = n+l-1$ and, for all $\sigma \in \mathcal{S}_k$:

$$r_{(k)\,\sigma(i_1)\ldots\sigma(i_k)} = \frac{c_j \alpha_{(j)\, i_1 \ldots i_j}}{k!} = \frac{j!}{k!} c_j a_{(j)\, i_1 \ldots i_j}$$

- otherwise $r_{(k)\, i_1\, \ldots\, i_k}$ is null.

[1] With the convention $[\![p, q]\!] = \emptyset$ if $p > q$

Layered e-adjacency hypermatrix

Choice of dilatation coefficient

We choose: $c_j = \frac{k_{\max}}{j}$ to allow the generalized hand-shake lemma to hold.

$$|E| = \frac{1}{k_{\max}} \sum_{i_1,\ldots,i_{k_{\max}} \in [\![n+k_{\max}-1]\!]} r_{i_1 \ldots i_{k_{\max}}} = \sum_{j=1}^{k_{\max}} \frac{1}{j} \sum_{i_1,\ldots,i_j \in [\![n]\!]} a_{(j)\, i_1 \ldots i_j}.$$

Hence, combining above with the fact that $a_{(j)\, i_1 \ldots i_j} = \frac{1}{(j-1)!}$ when $\left\{ v_{i_1}, \ldots, v_{i_j} \right\} \in E$ and 0 otherwise: $r_{i_1 \ldots i_{k_{\max}}} = \frac{1}{(k_{\max}-1)!}$ for nonzero elements of $\boldsymbol{R}_{\boldsymbol{k}_{\max}}$.

Layered e-adjacency hypermatrix

$\boldsymbol{R}_{\boldsymbol{k}_{\max}}$ is called the **layered e-adjacency tensor** of the hypergraph $\mathcal{H}$. We write it later $\mathcal{A}_{\mathcal{H}}$.

Properties

Finding degrees

It holds:

$$\sum_{\substack{i_2,\ldots,i_{k_{\max}}=1 \\ \delta_{ii_2\ldots i_{k_{\max}}}=0}}^{n+k_{\max}-1} a_{ii_2\ldots i_{k_{\max}}} = d_i$$

where: $\forall i \in [\![n]\!] : d_i = \deg(v_i)$ and $\forall i \in [\![k_{\max}-1]\!] : d_{n+i} = \deg(y_i)$.
Moreover: $\forall j \in [\![2; k_{\max}]\!]$:

$$|\{e : |e| = j\}| = d_{n+j} - d_{n+j-1}$$

and:

$$|\{e : |e| = 1\}| = d_{n+1}$$

Spectral results

Bound for eigenvalues

Theorem

The e-adjacency tensor $\mathcal{A}_{\mathcal{H}}$ has its eigenvalues λ such that:

$$|\lambda| \leqslant \max\left(\Delta, \Delta^{\star}\right) \tag{1}$$

where $\Delta = \max\limits_{1 \leqslant i \leqslant n} \left(d_i\right)$ *and* $\Delta^{\star} = \max\limits_{1 \leqslant i \leqslant k_{\max}-1} \left(d_{n+i}\right)$.

Theorem

Let $\mathcal{H}$ be a r-regular[2] r-uniform hypergraph with no repeated hyperedge. Then this maximum is reached.

[2] A hypergraph is said r-regular if all vertices have same degree r.

Conclusion

Quick summary and Future Work

- Layered e-adjacency tensor is easy to build
- Can be stored in $|E|$ elements as it is symmetric
- But inflates spectral bounds
- HUP and PUP: strong basis for further alternatives
- Target: allow repetition => **multisets are needed** => **hb-graphs introduced**

Ouvrard et al. [2018]

Thank you for your attention

Questions?

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